OLG: how to obtain eq. (10) and (11) in slide 44

— Week 7 —

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Remember slide 44

1. Profits are given by

$$\underset{K_t,L_t}{max}[F(K_t,L_t) - w_tL_t - r_tK_t].$$

2. First order conditions are

$$\begin{aligned} F'_K(K_t,L_t) &= r_t \\ F'_L(K_t,L_t) &= w_t \end{aligned}$$

3. As $F(K_t,L_t)$ is linear homogeneous the two conditions can be written as

$$f'(k_t) = r_t \quad (1)$$

$$f(k_t) - k_t \cdot f'(k_t) = w_t \quad (2)$$

4. Why $F'_L(K_t,L_t) = f(k_t) - k_t \cdot f'(k_t)$? And why $F'_K(K_t,L_t) = f'(k_t)$?

First you need to recall two differentiation rules

We need to recall the product rule and the chain rule in differentiation

The product rule in differentiation

The product rule is like this: for the functions $u(x)$ and $v(x)$, the derivative of the function $z(x) = u(x)v(x)$ with respect to x is

$$z(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x) \qquad \text{(Product rule)}$$

The Chain rule in differentiation

The chain rule tells us that if you have a function u which is a function of another function $v(x)$, the derivative of $z(x) = u[v(x)]$ with respect to x is

$$z'(x) = u'[v(x)] \cdot v'(x) \qquad \text{(Chain rule)}$$

Let's get back to our slides

1. Remember that total output is written as

$$Y_t = F(K_t, L_t) \tag{3}$$

2. Then, output per capita (y_t) is given by dividing the previous equation through by L_t

$$y_t = \frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = f(k_t)$$

3. Notice that

$$k_t = \frac{K_t}{L_t}$$

4. Therefore, total output can be written back as

$$Y_t = f(k_t) \cdot L_t \tag{4}$$

Let's apply the product rule

1. Notice that the previous equation can be written as

$$\begin{aligned} Y_t &= f(k_t) \cdot L_t \\ &= \underbrace{f\left(K_t/L_t\right)}_{h(L_t)} . \underbrace{L_t}_{g(L_t)} \\ &= h(L_t) \cdot g(L_t) \end{aligned} \tag{5}$$

2. Let's apply the product rule to eq. (5) **with respect to** L_t

$$\frac{dY_t}{dL_t} \text{ or } F_L'(K_t, L_t) \text{ or } Y_L'$$

which is given by

$$\begin{aligned} Y_L' &= \underbrace{g'(L_t)}_{1} \cdot \underbrace{h(L_t)}_{f(k_t)} + \underbrace{g(L_t)}_{L_t} \cdot \underbrace{h'(L_t)}_{chain\ rule} \\ &= 1 \cdot f(k_t) + L_t \cdot h'(L_t) \end{aligned} \tag{6}$$

Let's apply the chain rule

1. The chain rule is like this: if you have a function $z(x) = u[v(x)]$ then

$$z'(x) = u'[v(x)] \cdot v'(x)$$

2. Notice that our $h(L)$ function above was defined as

$$h(L) = h[f(k)] = h[f(K/L)]$$

3. So, let's apply the chain rule to $h(L)$

$$\begin{aligned} h'(L) &= \underbrace{h'[f(k)]}_{=-1\cdot\frac{K}{L^2}} \cdot f'(k) \\ &= -1 \cdot \frac{K}{L^2} \cdot f'(k) \qquad (7) \end{aligned}$$

4. Now we can insert this result back into eq.(6)

Putting the two rules together

1. With the product rule we got this result in eq.(6)

$$Y'_L = f(k_t) + L_t \cdot h'(L_t)$$

2. With the chain rule this result was obtained eq.(7)

$$h'(L) = -1 \cdot \frac{K}{L^2} \cdot f'(k)$$

3. Insert (7) into (6) and we will obtain

$$\begin{aligned} Y'_L &= 1 \cdot f(k_t) + L_t \cdot h'(L_t) \\ &= f(k_t) - \underbrace{\frac{K}{L}}_{k_t} \cdot f'(k) \end{aligned}$$

4. And the final result comes as

$$Y'_L = f(k_t) - k_t \cdot f'(k_t) \tag{8}$$

The marginal product of Capital is simpler

1. We know that total output can be written as

$$Y_t = f(k_t) \cdot L_t$$

2. Let's differentiate the previous equation **with respect to** K_t

$$\frac{dY_t}{dK_t} \text{ or } F'_K(K_t, L_t) \text{ or } Y'_K$$

3. If $F(K_t, L_t)$ is linear homogenous, it turns out that

$$Y'_K = f'(k_t) \tag{9}$$

4. There is no sophisticated differentiation rules to be applied here. Let's check this with an example next.

Example: the marginal product of capital

1. Assume we have a Cobb-Douglas production function (linear homogenous)
$$Y_t = \gamma K_t^{\alpha} L_t^{1-\alpha} \tag{10}$$
γ, α as parameters, Y is total output, K for total capital, L for labor.
2. Output per capita (y) is given by
$$y_t = \gamma k_t^{\alpha} \tag{11}$$
3. The marginal product of capital is given by differentiating eq.(10) with respect to K
$$Y'_K = \alpha\gamma K_t^{\alpha-1} L_t^{1-\alpha} = \alpha\gamma \frac{K_t^{\alpha-1}}{L_t^{\alpha-1}} = \alpha\gamma k_t^{\alpha-1}$$
4. Notice that if you use eq.(11) and apply directly the result in eq.(9), you will get the same result
$$Y'_K = f'(k_t) = \alpha\gamma k_t^{\alpha-1}.$$

Example: the marginal product of labor

1. The marginal product of labor is given by differentiating eq.(10) with respect to L

$$Y_L' = (1-\alpha)\gamma K_t^\alpha L_t^{1-\alpha-1} = (1-\alpha)\gamma \frac{K_t^\alpha}{L_t^\alpha} = (1-\alpha)\gamma k_t^\alpha$$

2. Notice that if you use eq.(11) and apply directly the result in eq.(8), you will get the same result

$$\begin{aligned} Y_K' &= f(k_t) - k_t \cdot f'(k_t) \\ &= \gamma k_t^\alpha - k_t \cdot \alpha\gamma k_t^{\alpha-1} \\ &= (1-\alpha)\gamma k_t^\alpha \end{aligned}$$

A final advice: do not worry

1. The **general results** (about linear homogenous production functions) presented on slide 44 in the original set of OLG slides are

$$F'_L(K_t, L_t) = f(k_t) - k_t \cdot f'(k_t)$$

$$F'_K(K_t, L_t) = f'(k_t)$$

2. The two examples above show that the calculus of $F'_L(K_t, L_t)$ and $F'_K(K_t, L_t)$ are in fact extremely easy to obtain ... **without the need to use these general results.**
3. So, just do not bother too much about those two general results. Economists like to generalize a lot. Sometimes too much.