Overlapping–Generations models
Problems and SOLUTIONS

Modern Macroeconomics — ISCTE–IUL — May 2014

Problem A. Using the Overlapping-Generations model, where c_t^y, c_{t+1}^o represent the level of consumption of one given generation, respectively, when it is young and when it is old, and c_1^o is the level of consumption of the first (initial) old generation, answer the following questions:

1. Define one allocation that satisfies the Pareto optimum.
2. Assume that the total revenue in each period is given by $TR = 20$. Assume also that the utility function of each generation is given by

$$u(c_t^y, c_{t+1}^o) = (3/4)\, c_t^y + c_{t+1}^o$$

   From the three following allocations, which one corresponds to the best possible allocation of resources over time?

   (a) Allocation A: $(c_t^y = 10, c_{t+1}^o = 10)$
   (b) Allocation B: $(c_t^y = 12, c_{t+1}^o = 8)$
   (c) Allocation C: $(c_t^y = 8, c_{t+1}^o = 12)$

3. Which tool can be used such that all allocations would satisfy the Pareto Optimum. Explain.

Solution A.

1. See slides for a definition:
"A Pareto optimal allocation is a sequence $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$ satisfying the constraint faced by the social planner and the property that there exists no other allocation $\{\hat{c}_t^y, \hat{c}_t^o, \hat{k}_{t+1}\}_{t=0}^{\infty}$ which satisfies such constraint and

$$\begin{aligned} \hat{c}_1^o &\geq c_1^o \\ u(\hat{c}_t^y, \hat{c}_{t+1}^o) &\geq u(c_t^y, c_{t+1}^o) \end{aligned}$$

for all $t = 0, 1, 2, 3, ...,$ with strict inequality in at least one instance."

2. The utility levels are all different:

$$\begin{aligned} u(c_t^y, c_{t+1}^o)_A &= (3/4)\,10 + 10 = 17.5 \\ u(c_t^y, c_{t+1}^o)_B &= (3/4)\,12 + 8 = 17 \\ u(c_t^y, c_{t+1}^o)_C &= (3/4)\,8 + 12 = 18 \end{aligned}$$

In terms of the welfare of the first (or initial) old generation we get

$$c_1^o(C) = 12 > c_1^o(A) = 10 > c_1^o(B) = 8.$$

Moreover,

$$u(c_t^y, c_{t+1}^o)_C > u(c_t^y, c_{t+1}^o)_A > u(c_t^y, c_{t+1}^o)_B$$

Therefore, Allocation C is superior to A, which in turn is superior to B.

3. The social planner could impose taxes upon the young generation (2 euros in A; 4 euros in B) and transfer that income to the old generation. In this case, all allocations would satisfy the Pareto optimum.

Problem B. Consider the following overlapping generations growth model. Time is indexed by $t = 0, 1, 2, ...$, and in period t there are L_t two-period-lived consumers born, where $L_1 = (1+n)^1 L_0$, with L_0 given and $n > 0$. In periods $t = 0, 1, 2, ...$, each young consumer is endowed with y units of the consumption good. Each old consumer (including the initial old in period 0) is endowed with nothing.

There is a given technology which permits one unit of period t consumption goods to be converted to $1+r$ units of period $t+1$ consumption goods, for $t = 0, 1, 2, ...$ In period t, the government collects a lump-sum tax of τ_t^y from each young consumer, and a lump-sum tax of τ_t^o from each old consumer.

A consumer born in period t has preferences given by

$$u(c_t^y, c_{t+1}^o) = 2(c_t^y)^{1/2} + 2(c_{t+1}^o)^{1/2}$$

1. Write down the government's budget constraint. (Hint: the only source of government revenue is the lump-sum taxes imposed upon each generation)

2. Suppose a *pay-as-you-go* social security scheme, where each period the government taxes the young so as to make transfers to the old. That is, the government sets $\tau_t^y = \tau$ for $t = 0, 1, 2, ...$ Determine the effects of an increase in τ on the savings of each young consumer, and on the welfare of each generation.

3. Determine also the Pareto optimal level of τ and explain your results. (Hint: this should depend whether $n > r$, or vice versa)

4. Suppose a *fully-funded* social security system where the government taxes the young in period t, puts the proceeds of the tax into storage, and then makes transfers to the old in period $t+1$ with the proceeds from period t storage. In this case the government sets $\tau_t^y = \tau$ for $t = 0, 1, 2, ...$ Determine the effects of an increase in τ on the savings of each young consumer, the consumption of the young and the old in each generation, and on the welfare of each generation.

5. As in question (3), determine the Pareto optimal level of τ and explain your results.

Solution B.

1. The government budget constraint at time t is

$$\tau_t^y L_t + \tau_t^o L_{t-1} = 0$$

Dividing through by L_t gives

$$\frac{\tau_t^y L_t}{L_t} + \frac{\tau_t^o L_{t-1}}{L_t} = 0$$

As $L_1 = (1+n)^1 L_0$, then

$$\frac{L_{t-1}}{L_t} = \frac{1}{1+n}$$

And we can obtain

$$\tau_t^y + \frac{\tau_t^o}{1+n} = 0 \tag{1}$$

2. A pay-as-you-go social security scheme. Notice that this is a simpler version of the problem as explained in the slides, because the government just taxes the young using a lump sum tax in each period of time. Notice also that the government transfers money from the young generation to the old one, but taxes also the old generation. These are the only differences from what was explained in the slides.

A PAYG depends upon the funding scheme. Following the slides, we can arrive at the following equation for the benefit per person (b) of the old generation

$$L_t \cdot \tau_t^y = b \cdot L_{t-1}$$

that is

$$b = (1+n)\tau_t^y$$

For each generation, the two constraints are as follows, considering (as we normally do) that savings at t are invested at $t+1$ at the rate r_{t+1} (for simplicity let's call r_{t+1} as r)

$$c_t^y + s_t = y_t - \tau_t^y \tag{2}$$

$$c_{t+1}^o = (1+r)s_t - \tau_{t+1}^o + \underbrace{(1+n)\tau_t^y}_{PAYG\_benefit} \tag{3}$$

Cancelling out s_t gives

$$c_t^y + \frac{c_{t+1}^o}{1+r} = y_t - \tau_t^y + \left(\frac{1+n}{1+r}\right)\tau_t^y - \frac{\tau_{t+1}^o}{1+r} \tag{4}$$

The Lagrangian will come as

$$\begin{aligned} L &= 2(c_t^y)^{1/2} + 2(c_{t+1}^o)^{1/2} + \\ &\quad \lambda\left[y_t - \tau_t^y + \left(\frac{1+n}{1+r}\right)\tau_t^y - \frac{\tau_{t+1}^o}{1+r} - c_t^y - \frac{c_{t+1}^o}{1+r}\right] \end{aligned}$$

Now take the first order conditions with respect to $c_t^y, c_{t+1}^o, \lambda$. From the first two we get (after eliminating λ) our already well known Euler equation (no discount factor here, so no β here)

$$(c_t^y)^{-1/2} = (1+r)\,(c_{t+1}^o)^{-1/2}$$

Notice that this equation can be further simplified as[1]

$$c_{t+1}^o = (1+r)^2 c_t^y$$

This Euler equation together with the intertemporal constraint (eq. 4) will give us the optimal level of consumption and savings

$$c_t^y = \frac{1}{2+r}\left[y_t - \tau_t^y + \left(\frac{1+n}{1+r}\right)\tau_t^y - \frac{\tau_{t+1}^o}{1+r}\right] \tag{5}$$

$$c_{t+1}^o = \frac{(1+r)^2}{2+r}\left[y_t - \tau_t^y + \left(\frac{1+n}{1+r}\right)\tau_t^y - \frac{\tau_{t+1}^o}{1+r}\right] \tag{6}$$

Now the optimal level of savings of the young generation can be obtained by using eq. (2) and eq. (5)

$$s_t = y_t - \tau_t^y - c_t^y$$

[1] Notice that $(c_t^y)^{-1/2} = (1+r)\,(c_{t+1}^o)^{-1/2}$ can be written as $(c_t^y/c_{t+1}^o)^{-1/2} = 1+r$. Therefore, $(c_{t+1}^o/c_t^y)^{1/2} = 1+r$, and finally by raising both sides by the power of 2, we get $(c_{t+1}^o/c_t^y) = (1+r)^2$.

Do it yourself (mere substitution).

To determine the effects of an increase in τ_t^y on the welfare of each generation and on the savings of each young consumer, we have just to calculate the first derivative with respect to τ_t^y. This is an easy task and I will just do this for the the impact of τ_t^y upon c_t^y : [2]

$$\frac{\partial c_t^y}{\partial \tau_t^y} = \frac{1}{2+r}\left[-1 + \left(\frac{1+n}{1+r}\right)\right] \tag{7}$$

This impact is

$$\begin{aligned} &= 0 \text{ if } n = r \\ &> 0 \text{ if } n > r \\ &< 0 \text{ if } n < r \end{aligned}$$

So, for example, if $n > r$ (population grows faster than the return from financial investments), and increase in taxes leads to higher c_t^y.

3. Let's exemplify our answer with the result in eq. (7). We know that if $n < r$ the impact of τ_t^y upon c_t^y is negative. Therefore in this case, the optimal level of τ_t^y would be zero, and so consumption should be the highest possible. So the results depend upon the relation between n and r:

$$\begin{aligned} \text{if } n &= r, \text{ the level of } \tau_t^y \text{ is irrelevant in this case} \\ \text{if } n &< r, \text{ the level of } \tau_t^y \text{ should be zero} \\ \text{if } n &> r, \text{ the level of } \tau_t^y \text{ should be as higher as possible} \end{aligned}$$

4. A fully-funded social security scheme. In this case the two period budget constraints are

$$c_t^y + s_t = y_t - \tau_t^y \tag{8}$$

$$c_{t+1}^o = (1+r)s_t - \tau_{t+1}^o + \underbrace{(1+r)\tau_t^y}_{Funded\_benefit} \tag{9}$$

and the intertemporal budget constraint will come as

$$\begin{aligned} c_t^y + \frac{c_{t+1}^o}{1+r} &= y_t - \tau_t^y + \left(\frac{1+r}{1+r}\right)\tau_t^y - \frac{\tau_{t+1}^o}{1+r} \\ &= y_t - \frac{\tau_{t+1}^o}{1+r} \end{aligned} \tag{10}$$

The Lagrangian will come out as

[2] Try yourself the impacts of τ_t^y upon c_t^y and s_t.

$$L = 2(c_t^y)^{1/2} + 2(c_{t+1}^o)^{1/2} + \lambda \left[y_t - \frac{\tau_{t+1}^o}{1+r} - c_t^y - \frac{c_{t+1}^o}{1+r} \right]$$

Performing the same steps as in question 2, we will obtain the Euler equation

$$c_{t+1}^o = (1+r)^2 c_t^y$$

and together with eq. (10), we will obtain the optimal levels of consumption

$$\begin{aligned} c_t^y &= \frac{1}{2+r} \left[y_t - \frac{\tau_{t+1}^o}{1+r} \right] \\ c_{t+1}^o &= \frac{(1+r)^2}{2+r} \left[y_t - \frac{\tau_{t+1}^o}{1+r} \right] \end{aligned}$$

and you should calculate the solution for s_t.

Now it is very easy to see that the taxes upon the young (τ_t^y) will not affect the optimal levels of consumption, but it will affect the level of savings (see eq. 8). However, the level of utility (which depends upon c_t^y and c_{t+1}^o) will be dependent upon the level of taxes paid by the old generation (τ_{t+1}^o). For example,

$$\frac{\partial c_t^y}{\partial \tau_{t+1}^o} = -\frac{1}{2+r} \left(\frac{1}{1+r} \right)$$

which is always negative (the same for $\frac{\partial c_{t+1}^o}{\partial \tau_{t+1}^o}$).

5. Given these results, the optimal level of taxes upon the old generation (τ_{t+1}^o) should be zero in this case, in order to maximize intertemporal welfare..