Solutions : Two Period Economies

ISCTE - IUL

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— Modern Macroeconomics —

(19 pages)

<u>Problem 1.</u>

The intertemporal consolidated budget constraint is given by

$$c_t + \frac{c_{t+1}}{1+r_{t+1}} = w_t + \frac{w_{t+1}}{1+r_{t+1}} \qquad (1)$$

Therefore the Lagrangian can be written as

$$\mathcal{L} = u(c_t) + \beta u(c_{t+1}) + \lambda_t \left[w_t + \frac{w_{t+1}}{1+r_{t+1}} - c_t - \frac{c_{t+1}}{1+r_{t+1}} \right]$$

The First Order Conditions are given by

$$\partial \mathcal{L} / \partial c_t = 0 \Rightarrow u'(c_t) - \lambda_t = 0 \qquad (2)$$

$$\partial \mathcal{L} / \partial c_{t+1} = 0 \Rightarrow \beta u'(c_{t+1}) - \frac{\lambda_t}{1+r_{t+1}} = 0 \qquad (3)$$

$$\partial \mathcal{L} / \partial \lambda_t = 0 \Rightarrow w_t + \frac{w_{t+1}}{1+r_{t+1}} - c_t - \frac{c_{t+1}}{1+r_{t+1}} = 0 \qquad (4)$$

Now from (2) and (3) we can obtain

$$u'(c_t) = \lambda_t$$

$$\beta u'(c_{t+1})(1+r_{t+1}) = \lambda_t$$

from which it is easy to get

$$u'(c_t) = \beta(1+r_{t+1})\, u'(c_{t+1}) \qquad (5)$$

Equation (5) is the well known Euler Equation, and is given by exactly the same terms as in the two period sequential decision making process that was used in classes.

From now onwards the results are exactly the same. To obtain the optimal levels of c_t and c_{t+1} (c_t^*, c_{t+1}^*) we should use eq. (5) and (1). And after the level of c_t^* is known we can obtain the optimal level of savings (a_{t+1}^*) just by using the period 1 budget constraint: $c_t + a_{t+1} = w_t$.

Problem 2

1. $$u'(c_t) = \frac{\partial \ln c_t}{\partial c_t} = \frac{1}{c_t} > 0 \qquad (6)$$

$$u''(c_t) = \frac{\partial^2 \ln c_t}{\partial c_t^2} = -\frac{1}{c_t^2} < 0$$

and the same applies to the derivatives with respect to c_{t+1}. Therefore the conditions are satisfied.

2. The Euler equation is given by

$$u'(c_t) = \beta(1+r_{t+1})\, u'(c_{t+1}) \qquad (7)$$

Inserting the results in (6) into (7) we get

$$\frac{1}{c_t} = \beta(1+r_{t+1})\frac{1}{c_{t+1}}$$

That is

$$c_{t+1} = \beta(1+r_{t+1})\, c_t \qquad (8)$$

3. The solutions to the optimal values of c_t and c_{t+1} (c_t^* and c_{t+1}^*) are obtained when they are totally determined only by parameters and by exogenous variables. From equation (8) we know that $c_{t+1} = f(c_t)$ and from the intertemporal consolidated budget constraint

$$c_t + \frac{c_{t+1}}{1+r_{t+1}} = w_t + \frac{w_{t+1}}{1+r_{t+1}} \qquad (9)$$

we get another relationship between c_t and c_{t+1}. Therefore, using eq. (8) and (9), we have two unknowns and two equations, and we can obtain the values we are looking for. Notice that (9) can be written as

$$c_t = w_t + \frac{w_{t+1}}{1+r_{t+1}} - \frac{c_{t+1}}{1+r_{t+1}} . \qquad (10)$$

Now insert the result in (10) into eq. (8) and

$$c_{t+1} = \beta(1+r_{t+1})\left[w_t + \frac{w_{t+1}}{1+r_{t+1}} - \frac{c_{t+1}}{1+r_{t+1}}\right]$$

from where we can obtain

$$c_{t+1} = \beta(1+r_{t+1})w_t + \beta w_{t+1} - \beta c_{t+1}$$

and the optimal value for c_{t+1} (that is, c^*_{t+1}) is finally given by solving the previous equation for c_{t+1}

$$c^*_{t+1} = \frac{\beta}{1+\beta}\left[w_{t+1} + (1+r_{t+1})w_t\right]. \quad (11)$$

Now, it is easy to obtain the optimal value for c_t. Using eq. (8), we can obtain

$$c^*_t = \frac{1}{\beta(1+r_{t+1})} c^*_{t+1} \qquad (12)$$

and inserting the optimal value of c^*_{t+1} in (11) into (12), and after some rearrangements we get

$$c^*_t = \frac{1}{1+\beta}\left[w_t + \frac{w_{t+1}}{1+r_{t+1}}\right].$$

(4) Confirm that c_t^* and c_{t+1}^* do really satisfy the intertemporal constraint. This is just a time consuming exercise. From (9)

Don't do it: only this question

$$c_t^* + \frac{c_{t+1}^*}{1+r_{t+1}} = w_t + \frac{w_{t+1}}{1+r_{t+1}}$$

and in order to simplify, denominate $1+r_{t+1} = \phi$.

$$c_t^* + \frac{c_{t+1}}{\phi} = w_t + \frac{w_{t+1}}{\phi}$$

$$\frac{1}{1+\beta}\left[w_t + \frac{w_{t+1}}{\phi}\right] + \frac{1}{\phi}\frac{\beta}{1+\beta}\left[w_{t+1} + \phi w_t\right] = w_t + \frac{w_{t+1}}{\phi}$$

simplifying, by subtracting the first term on the left hand side from both sides and rearranging

$$\frac{1}{\phi}\frac{\beta}{1+\beta}w_{t+1} + \frac{\beta}{1+\beta}w_t = \left(w_t + \frac{w_{t+1}}{\phi}\right)\left(1 - \frac{1}{1+\beta}\right)$$

$$\frac{1}{\phi}\frac{\beta}{1+\beta}w_{t+1} + \frac{\beta}{1+\beta}w_t = \left(w_t + \frac{w_{t+1}}{\phi}\right)\frac{\beta}{1+\beta}$$

$$\frac{1}{\phi}\frac{\beta}{1+\beta}w_{t+1} + \frac{\beta}{1+\beta}w_t = \frac{\beta}{1+\beta}w_t + \frac{1}{\phi}\frac{\beta}{1+\beta}w_{t+1}.$$

(5) What conditions have to be satisfied such that $a_{t+1} < 0$? We know that from the constraint at t

$$c_t + a_{t+1} = w_t$$

$$a^*_{t+1} = w_t - c^*_t$$

$$a^*_{t+1} = w_t - \frac{1}{1+\beta}\left(w_t + \frac{w_{t+1}}{1+r_{t+1}}\right)$$

$$a^*_{t+1} = w_t - \frac{1}{1+\beta} w_t - \frac{1}{1+\beta}\frac{w_{t+1}}{1+r_{t+1}}$$

$$a^*_{t+1} = \frac{\beta}{1+\beta} w_t - \frac{1}{(1+\beta)(1+r_{t+1})} w_{t+1}$$

$$a^*_{t+1} = \frac{1}{1+\beta}\left(\beta w_t - \frac{1}{1+r_{t+1}} w_{t+1}\right).$$

Therefore, in order to have $a^*_{t+1} < 0$, the following condition has to be satisfied

$$\beta w_t < \frac{1}{1+r_{t+1}} w_{t+1}.$$

Problem 3. This problem is totally similar to the previous one. The only difference now is that in the current problem we have a different utility function

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

1. Not to be solved by undergraduate students.

3. We can answer question number 3 straightaway. The first derivative of $u(c_t)$ with respect to c_t is given by

$$u'(c_t) = \frac{\partial u(c_t)}{\partial c_t} = \frac{(1-\sigma)\, c_t^{1-\sigma-1}}{1-\sigma} =$$

$$= c_t^{-\sigma} = \frac{1}{c_t^{\sigma}}.$$

Therefore, when $\sigma = 1$, we get the same results as in the case with logarithmic utility, where as you may remember, if $u(c_t) = \ln c_t$, then

$$u'(c_t) = \frac{1}{c_t}.$$

2. In order to derive the optimal values for c_t and c_{t+1} (c_t^*, c_{t+1}^*) we have firstly to write down the Euler equation for this particular form of utility. If the Euler equation is given by

$$u'(c_t) = \beta(1+r_{t+1})\, u'(c_{t+1})$$

for this particular form of utility we get

$$\frac{1}{c_t^\sigma} = \beta(1+r_{t+1})\frac{1}{c_{t+1}^\sigma}$$

$$\frac{c_{t+1}^\sigma}{c_t^\sigma} = \beta(1+r_{t+1})$$

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = \beta(1+r_{t+1})$$

$$\left[\left(\frac{c_{t+1}}{c_t}\right)^\sigma\right]^{\frac{1}{\sigma}} = \left[\beta(1+r_{t+1})\right]^{\frac{1}{\sigma}}$$

From where we can obtain

$$\left(\frac{c_{t+1}}{c_t}\right)^{\frac{\sigma}{\sigma}} = \left[\beta(1+r_{t+1})\right]^{\frac{1}{\sigma}}$$

$$\frac{c_{t+1}}{c_t} = \left[\beta(1+r_{t+1})\right]^{\frac{1}{\sigma}}$$

$$c_{t+1} = \left[\beta(1+r_{t+1})\right]^{1/\sigma} . c_t . \qquad (13)$$

Now, for simplicity, call the term above $\left[\beta(1+r_{t+1})\right]^{1/\sigma}$ by a simple letter like

$$\left[\beta(1+r_{t+1})\right]^{\frac{1}{\sigma}} = z$$

and the Euler equation will be

$$c_{t+1} = z . c_t . \qquad (14)$$

From now onwards, we should proceed exactly with the same steps as in the previous problem.

Problem 4. This is a problem that is totally similar to the two previous problems, as well. The only difference is that in the current problem we have a consumption tax (τ). This tax affects the intertemporal consolidated budget constraint as follows

$$(1-\tau)c_t + \frac{(1-\tau)c_{t+1}}{1+r_{t+1}} = w_t + \frac{w_{t+1}}{1+r_{t+1}}$$

If you want to practice try to solve this exercise assuming an utility function as in problem 2: $u(c_t) = \ln c_t$.

Notice that the Euler equation is the same as in problem 2 because the term $(1-\tau)$ cancell out. Try it:

$$\mathcal{L} = u(c_t) + \beta u(c_{t+1}) +$$

$$\lambda_t \left[w_t + \frac{w_{t+1}}{1+r_{t+1}} - (1-\tau) c_t - \frac{(1-\tau) c_{t+1}}{1+r_{t+1}} \right] \quad (15)$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \quad u'(c_t) - (1-\tau)\lambda_t = 0 \qquad (16)$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \Rightarrow \beta u'(c_t) - \frac{(1-\tau)}{1+r_{t+1}} \lambda_t = 0 \qquad (17)$$

Solving (16) and (17) for λ_t and equalizing both equations, we will get

$$u'(c_t) = \beta (1+r_{t+1}) u'(c_{t+1}) \, ,$$

which is exactly equal to our well know Euler equation.

From now onwards, proceed as usual.

<u>Problem 5</u>. This is a good exercise because it deals with liquidity constraints. From the information available we know that

$$u(c_t, c_{t+1}) = c_t \cdot c_{t+1}^{\alpha}$$

$$r_{t+1} = 11\%$$

$$w_t = w_{t+1} = 10$$

$$\alpha = 0.5$$

1. In order to further simplify things, let us assume that the intertemporal discount rate (our β) is equal to 1. That is, assume

$$\beta = 1.$$

Therefore the Euler equation comes out as

$$u'(c_t) = (1 + r_{t+1})\, u'(c_{t+1}).$$

Therefore, just by applying the information we get from the exercise, the Euler equation comes as

$$u'(c_t) = (1+r_{t+1})\, u'(c_{t+1})$$

$$c_{t+1}'^{\alpha} = (1+r_{t+1})\, \alpha\, c_t\, c_{t+1}'^{\alpha-1}$$

$$c_{t+1} = (1+r_{t+1})\, 0.5\, c_t$$

$$c_{t+1} = (1.11)\, 0.5\, c_t$$

$$c_{t+1} = 0.555\, c_t \,. \qquad (18)$$

The consolidated intertemporal budget constraint is given by

$$w_t + \frac{w_{t+1}}{1+r_{t+1}} = c_t + \frac{c_{t+1}}{1+r_{t+1}}$$

$$10 + \frac{10}{1.11} = c_t + \frac{c_{t+1}}{1.11}$$

$$c_t = 10 + \frac{10}{1.11} - \frac{c_{t+1}}{1.11} \,. \qquad (19)$$

Next, just by using (18) and (19), we get

$$c^*_{t+1} = 0.555\left[10 + \frac{10}{1.11} - \frac{c^*_{t+1}}{1.11}\right]$$

from where we get

$$c^*_{t+1} = \frac{10.55}{1.5} = 7.$$

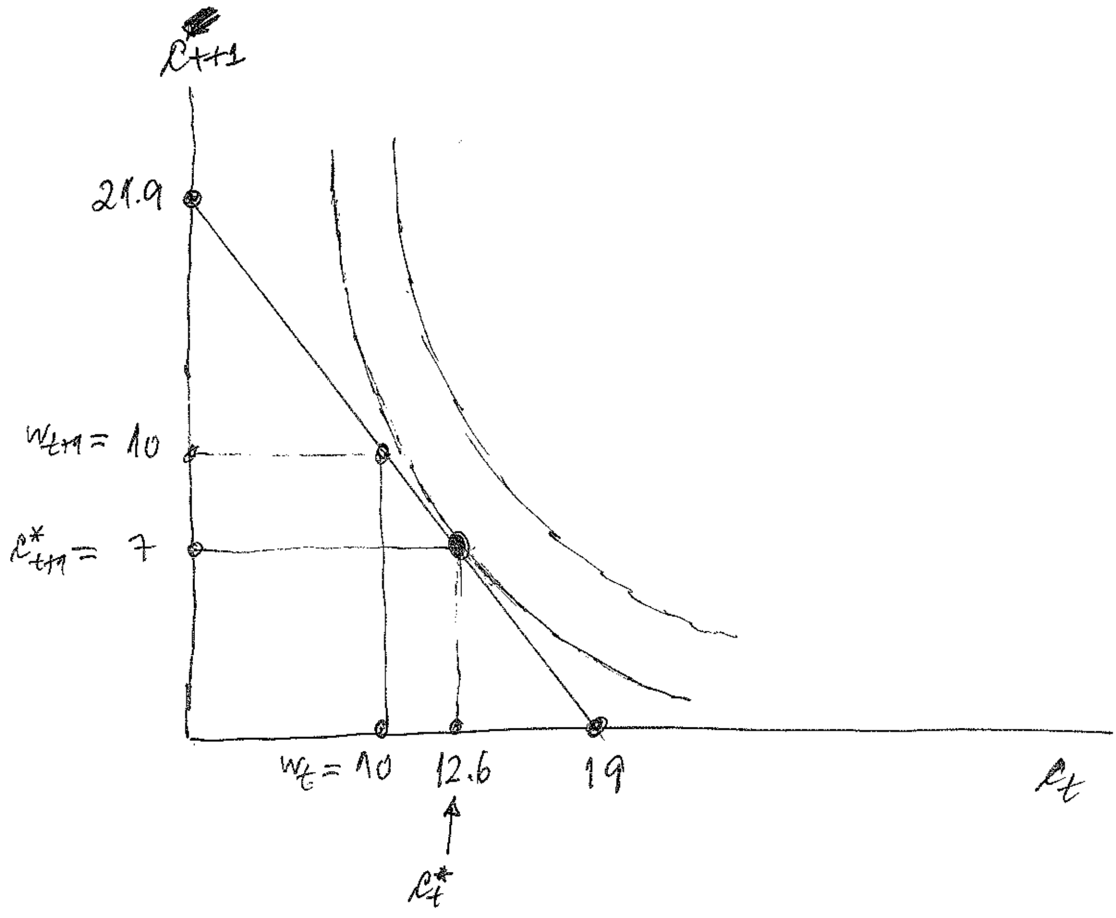
The result for c^*_t is immediate

$$c^*_t = \frac{c^*_{t+1}}{0.555} = \frac{7}{0.555} = 12.6.$$

Once we have the results for c^*_t and c^*_{t+1}, the level for a^*_{t+1} is straightforward

$$\begin{aligned} a^*_{t+1} &= w_t - c^*_t \\ &= 10 - 12.6 \\ &= -2.6. \end{aligned}$$

The graphical solution can now be presented.



In order to obtain the corner values ($c_t = 19$, $c_{t+1} = 0$) and ($c_t = 0$, $c_{t+1} = 21.9$) use the intertemporal consolidated constraint and impose the appropriate conditions.

(2) Yes, the liquidity contraint $a_{t+1} \geq -1$ does affect the solution in (1), because we got there a value for a_{t+1} much more negative than just -1. Remember that in (1) we got

$$a^*_{t+1} = -2.6 .$$

Therefore, if there were no liquidity constraints, our agent could borrow an amount A that is in this case

$$A = 2.6 .$$

What happens if in our present case, the private agent cannot borrow more than $A = 1$? The answer is: <u>the liquidity constraints is binding now, because she wants to borrow 2.6, but she is allowed only 1.</u>

In this case, the solution is given by

$$c^*_t = w_t + A = 10 + 1 = 11$$

and

$$c^*_{t+1} = w_{t+1} - (1+r_{t+1})A$$

$$= 10 - 1.11$$

$$= 8.89$$

3. If $a_{t+1} = +1$, then this implies that our agent is in fact forced to have positive savings, not a borrower anymore. This amount of savings is equal to 1. In this case, we have here a third possibility in our exercise. Even if she does not want to have positive savings, she is forced to do so. In this case

$$a^*_{t+1} = +1 = -A$$

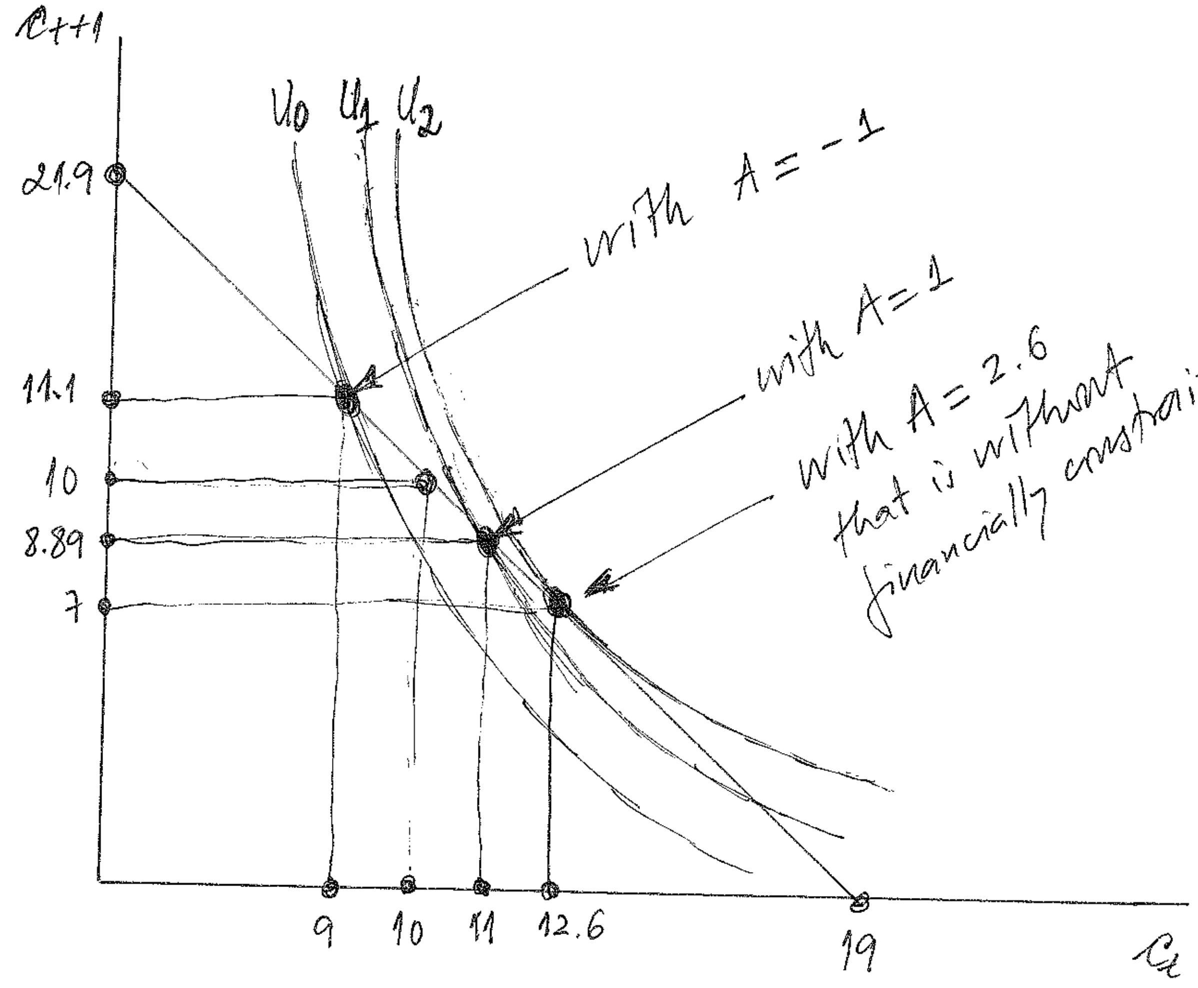
$$A = -1.$$

Assuming $A = -1$, we get the optimal values

$$c^*_t = w_t + A = 10 - 1 = 9$$

$$c^*_{t+1} = w_{t+1} - (1+r_{t+1})A = 11.1$$

We can now represent graphically the three cases involved in this exercise.



Notice that in terms of welfare the best solution is given by U_2, and the worst case is that where the consumer is forced to have positive savings ($a_{t+1} = 1$).