Notes on Macroeconomic Theory

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Read only sections:
2.1 + 2.2 + 2.3 + 2.4 + 2.6,
that is
from page 23 to 31.

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Chapter 2

Growth With Overlapping Generations

This chapter will serve as an introduction to neoclassical growth theory and to the overlapping generations model. The particular model introduced in this chapter was developed by Diamond (1965), building on the overlapping generations construct introduced by Samuelson (1956). Samuelson's paper was a semi-serious (meaning that Samuelson did not take it too seriously) attempt to model money, but it has also proved to be a useful vehicle for studying public finance issues such as government debt policy and the effects of social security systems. There was a resurgence in interest in the overlapping generations model as a monetary paradigm in the late seventies and early eighties, particularly at the University of Minnesota (see for example Kareken and Wallace 1980).

A key feature of the overlapping generations model is that markets are incomplete, in a sense, in that economic agents are finite-lived, and agents currently alive cannot trade with the unborn. As a result, competitive equilibria need not be Pareto optimal, and Ricardian equivalence does not hold. Thus, the timing of taxes and the size of the government debt matters. Without government intervention, resources may not be allocated optimally among generations, and capital accumulation may be suboptimal. However, government debt policy can be used as a vehicle for redistributing wealth among generations and inducing optimal savings behavior.

2.1 The Model

This is an infinite horizon model where time is indexed by $t = 0, 1, 2, ..., \infty$. Each period, L_t two-period-lived consumers are born, and each is endowed with one unit of labor in the first period of life, and zero units in the second period. The population evolves according to

$$L_t = L_0(1+n)^t, \tag{2.1}$$

where L_0 is given and $n > 0$ is the population growth rate. In period 0 there are some old consumers alive who live for one period and are collectively endowed with K_0 units of capital. Preferences for a consumer born in period t, $t = 0, 1, 2, ...$, are given by

$$u(c_t^y, c_{t+1}^o),$$

where c_t^y denotes the consumption of a young consumer in period t and c_t^o is the consumption of an old consumer. Assume that $u(\cdot, \cdot)$ is strictly increasing in both arguments, strictly concave, and defining

$$v(c^y, c^o) \equiv \frac{\frac{\partial u}{\partial c^y}}{\frac{\partial u}{\partial c^o}},$$

assume that $\lim_{c^y \to o} v(c^y, c^o) = \infty$ for $c^o > 0$ and $\lim_{c^o \to o} v(c^y, c^o) = 0$ for $c^y > 0$. These last two conditions on the marginal rate of substitution will imply that each consumer will always wish to consume positive amounts when young and when old. The initial old seek to maximize consumption in period 0.

The investment technology works as follows. Consumption goods can be converted one-for-one into capital, and vice-versa. Capital constructed in period t does not become productive until period $t+1$, and there is no depreciation.

Young agents sell their labor to firms and save in the form of capital accumulation, and old agents rent capital to firms and then convert the capital into consumption goods which they consume. The representative firm maximizes profits by producing consumption goods, and renting capital and hiring labor as inputs. The technology is given by

$$Y_t = F(K_t, L_t),$$

where Y_t is output and K_t and L_t are the capital and labor inputs, respectively. Assume that the production function $F(\cdot,\cdot)$ is strictly increasing, strictly quasi-concave, twice differentiable, and homogeneous of degree one.

2.2 Optimal Allocations

As a benchmark, we will first consider the allocations that can be achieved by a social planner who has control over production, capital accumulation, and the distribution of consumption goods between the young and the old. We will confine attention to allocations where all young agents in a given period are treated identically, and all old agents in a given period receive the same consumption.

The resource constraint faced by the social planner in period t is

$$F(K_t, L_t) + K_t = K_{t+1} + c_t^y L_t + c_t^o L_{t-1}, \tag{2.2}$$

where the left hand side of (2.2) is the quantity of goods available in period t, i.e. consumption goods produced plus the capital that is left after production takes place. The right hand side is the capital which will become productive in period $t+1$ plus the consumption of the young, plus consumption of the old.

In the long run, this model will have the property that per-capita quantities converge to constants. Thus, it proves to be convenient to express everything here in per-capita terms using lower case letters. Define $k_t \equiv \frac{K_t}{L_t}$ (the capital/labor ratio or per-capita capital stock) and $f(k_t) \equiv F(k_t, 1)$. We can then use (2.1) to rewrite (2.2) as

$$f(k_t) + k_t = (1+n)k_{t+1} + c_t^y + \frac{c_t^o}{1+n} \tag{2.3}$$

Definition 1 *A Pareto optimal allocation is a sequence* $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$ *satisfying (2.3) and the property that there exists no other allocation* $\{\hat{c}_t^y, \hat{c}_t^o, \hat{k}_{t+1}\}_{t=0}^{\infty}$ *which satisfies (2.3) and*

$$\hat{c}_1^o \geq c_1^o$$

$$u(\hat{c}_t^y, \hat{c}_{t+1}^o) \geq u(c_t^y, c_{t+1}^o)$$

for all $t = 0, 1, 2, 3, ...,$ *with strict inequality in at least one instance.*

That is, a Pareto optimal allocation is a feasible allocation such that there is no other feasible allocation for which all consumers are at least as well off and some consumer is better off. While Pareto optimality is the appropriate notion of social optimality for this model, it is somewhat complicated (for our purposes) to derive Pareto optimal allocations here. We will take a shortcut by focusing attention on steady states, where $k_t = k$, $c_t^y = c^y$, and $c_t^o = c^o$, where k, c^y, and c^o are constants. We need to be aware of two potential problems here. First, there may not be a feasible path which leads from k_0 to a particular steady state. Second, one steady state may dominate another in terms of the welfare of consumers once the steady state is achieved, but the two allocations may be Pareto non-comparable along the path to the steady state.

The problem for the social planner is to maximize the utility of each consumer in the steady state, given the feasibility condition, (2.2). That is, the planner chooses c^y, c^o, and k to solve

$$\max u(c^y, c^o)$$

subject to

$$f(k) - nk = c^y + \frac{c_o}{1+n}. \tag{2.4}$$

Substituting for c^o in the objective function using (2.4), we then solve the following

$$\max_{c^y,k} u(c^y, [1+n][f(k) - nk - c^y])$$

The first-order conditions for an optimum are then

$$u_1 - (1+n)u_2 = 0,$$

or

$$\frac{u_1}{u_2} = 1 + n \tag{2.5}$$

(intertemporal marginal rate of substitution equal to $1 + n$) and

$$f'(k) = n \tag{2.6}$$

(marginal product of capital equal to n). Note that the planner's problem splits into two separate components. First, the planner finds the capital-labor ratio which maximizes the steady state quantity of resources, from (2.6), and then allocates consumption between the young and the old according to (2.5). In Figure 2.1, k is chosen to maximize the size of the budget set for the consumer in the steady state, and then consumption is allocated between the young and the old to achieve the tangency between the aggregate resource constraint and an indifference curve at point A.

2.3 Competitive Equilibrium

In this section, we wish to determine the properties of a competitive equilibrium, and to ask whether a competitive equilibrium achieves the steady state social optimum characterized in the previous section.

2.3.1 Young Consumer's Problem

A consumer born in period t solves the following problem.

$$\max_{c_t^y, c_{t+1}^o, s_t} u(c_t^y, c_{t+1}^o)$$

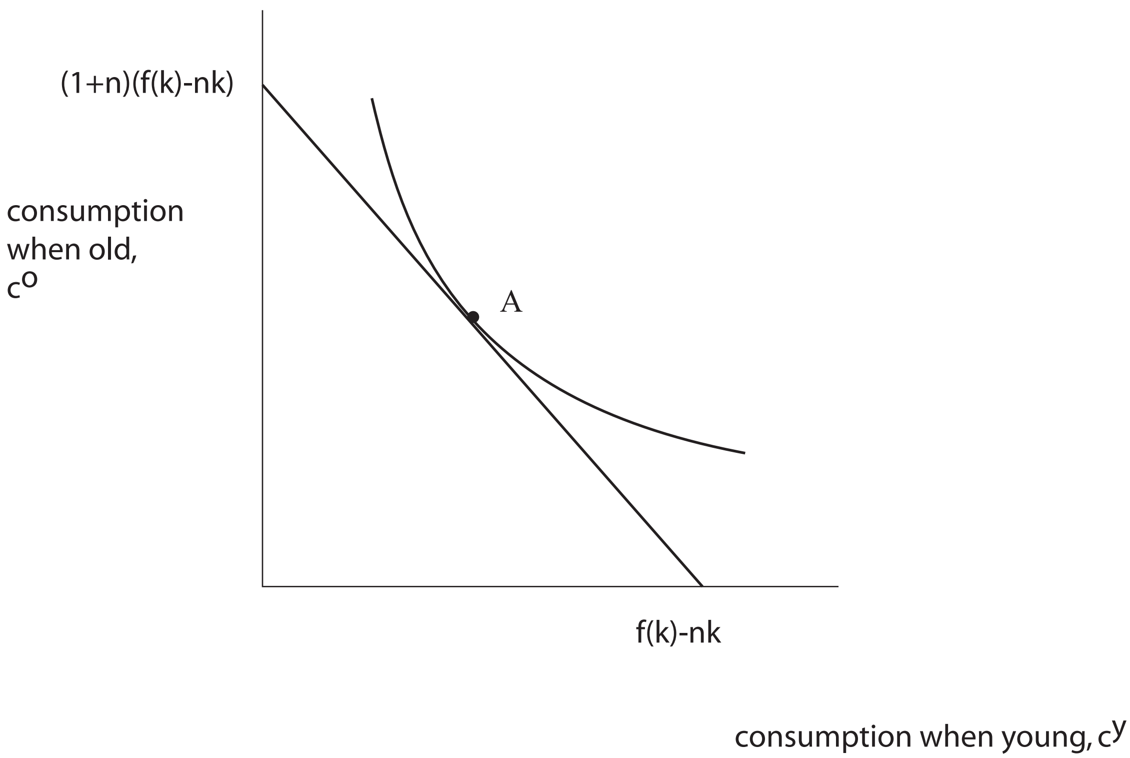
subject to

$$c_t^y = w_t - s_t \tag{2.7}$$

$$c_{t+1}^o = s_t(1 + r_{t+1}) \tag{2.8}$$

Here, w_t is the wage rate, r_t is the capital rental rate, and s_t is saving when young. Note that the capital rental rate plays the role of an interest rate here. The consumer chooses savings and consumption when young and old treating prices, w_t and r_{t+1}, as being fixed. At time t the consumer is assumed to know r_{t+1}. Equivalently, we can think of this as a rational expectations or perfect foresight equilibrium, where each consumer forecasts future prices, and optimizes based on those forecasts. In equilibrium, forecasts are correct, i.e. no one makes systematic forecasting errors. Since there is no uncertainty here, forecasts cannot be incorrect in equilibrium if agents have rational expectations.

Figure 2.1: Optimal Steady State in the OG Model



Substituting for c_t^y and c_{t+1}^o in the above objective function using (2.7) and (2.8) to obtain a maximization problem with one choice variable, s_t, the first-order condition for an optimum is then

$$-u_1(w_t - s_t, s_t(1+r_{t+1})) + u_2(w_t - s_t, s_t(1+r_{t+1}))(1+r_{t+1}) = 0 \quad (2.9)$$

which determines s_t, i.e. we can determine optimal savings as a function of prices

$$s_t = s(w_t, r_{t+1}). \quad (2.10)$$

Note that (2.9) can also be rewritten as $\frac{u_1}{u_2} = 1 + r_{t+1}$, i.e. the intertemporal marginal rate of substitution equals one plus the interest rate. Given that consumption when young and consumption when old are both normal goods, we have $\frac{\partial s}{\partial w_t} > 0$, however the sign of $\frac{\partial s}{\partial r_{t+1}}$ is indeterminate due to opposing income and substitution effects.

2.3.2 Representative Firm's Problem

The firm solves a static profit maximization problem

$$\max_{K_t, L_t} [F(K_t, L_t) - w_t L_t - r_t K_t].$$

The first-order conditions for a maximum are the usual marginal conditions

$$F_1(K_t, L_t) - r_t = 0,$$

$$F_2(K_t, L_t) - w_t = 0.$$

Since $F(\cdot, \cdot)$ is homogeneous of degree 1, we can rewrite these marginal conditions as

$$f'(k_t) - r_t = 0, \quad (2.11)$$

$$f(k_t) - k_t f'(k_t) - w_t = 0. \quad (2.12)$$

2.3.3 Competitive Equilibrium

Definition 2 *A competitive equilibrium is a sequence of quantities,* $\{k_{t+1}, s_t\}_{t=0}^{\infty}$ *and a sequence of prices* $\{w_t, r_t\}_{t=0}^{\infty}$*, which satisfy (i) consumer optimization; (ii) firm optimization; (iii) market clearing; in each period* $t = 0, 1, 2, ...,$ *given the initial capital-labor ratio* k_0.

Here, we have three markets, for labor, capital rental, and consumption goods, and Walras' law tells us that we can drop one market-clearing condition. It will be convenient here to drop the consumption goods market from consideration. Consumer optimization is summarized by equation (2.10), which essentially determines the supply of capital, as period t savings is equal to the capital that will be rented in period $t+1$. The supply of labor by consumers is inelastic. The demands for capital and labor are determined implicitly by equations (2.11) and (2.12). The equilibrium condition for the capital rental market is then

$$k_{t+1}(1+n) = s(w_t, r_{t+1}), \tag{2.13}$$

and we can substitute in (2.13) for w_t and r_{t+1} from (2.11) and (2.12) to get

$$k_{t+1}(1+n) = s(f(k_t) - kf'(k_t), f'(k_{t+1})). \tag{2.14}$$

Here, (2.14) is a nonlinear first-order difference equation which, given k_0, solves for $\{k_t\}_{t=1}^{\infty}$. Once we have the equilibrium sequence of capital-labor ratios, we can solve for prices from (2.11) and (2.12). We can then solve for $\{s_t\}_{t=0}^{\infty}$ from (2.10), and in turn for consumption allocations.

2.4 An Example

Let $u(c^y, c^o) = \ln c^y + \beta \ln c^o$, and $F(K, L) = \gamma K^{\alpha} L^{1-\alpha}$, where $\beta > 0$, $\gamma > 0$, and $0 < \alpha < 1$. Here, a young agent solves

$$\max_{s_t} [\ln(w_t - s_t) + \beta \ln[(1 + r_{t+1})s_t)],$$

and solving this problem we obtain the optimal savings function

$$s_t = \frac{\beta}{1+\beta} w_t. \tag{2.15}$$

Given the Cobb-Douglass production function, we have $f(k) = \gamma k^{\alpha}$ and $f'(k) = \gamma \alpha k^{\alpha-1}$. Therefore, from (2.11) and (2.12), the first-order conditions from the firm's optimization problem give

$$r_t = \gamma \alpha k_t^{\alpha-1}, \tag{2.16}$$

$$w_t = \gamma(1-\alpha)k_t^{\alpha}. \tag{2.17}$$

Then, using (2.14), (2.15), and (2.17), we get

$$k_{t+1}(1+n) = \frac{\beta}{(1+\beta)}\gamma(1-\alpha)k_t^{\alpha}. \tag{2.18}$$

Now, equation (2.18) determines a unique sequence $\{k_t\}_{t=1}^{\infty}$ given k_0 (see Figure 2m) which converges in the limit to k^*, the unique steady state capital-labor ratio, which we can determine from (2.18) by setting $k_{t+1} = k_t = k^*$ and solving to get

$$k^* = \left[\frac{\beta\gamma(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}. \tag{2.19}$$

Now, given the steady state capital-labor ratio from (2.19), we can solve for steady state prices from (2.16) and (2.17), that is

$$r^* = \frac{\alpha(1+n)(1+\beta)}{\beta(1-\alpha)},$$

$$w^* = \gamma(1-\alpha)\left[\frac{\beta\gamma(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{\alpha}{1-\alpha}}.$$

We can then solve for steady state consumption allocations,

$$c^y = w^* - \frac{\beta}{1+\beta}w^* = \frac{w^*}{1+\beta},$$

$$c^o = \frac{\beta}{1+\beta}w^*(1+r^*).$$

In the long run, this economy converges to a steady state where the capital-labor ratio, consumption allocations, the wage rate, and the rental rate on capital are constant. Since the capital-labor ratio is constant in the steady state and the labor input is growing at the rate n, the growth rate of the aggregate capital stock is also n in the steady state. In turn, aggregate output also grows at the rate n.

Now, note that the socially optimal steady state capital stock, $\hat{k}$, is determined by (2.6), that is

$$\gamma\alpha\hat{k}^{\alpha-1} = n,$$

or

$$\hat{k} = \left(\frac{\alpha\gamma}{n}\right)^{\frac{1}{1-\alpha}}. \tag{2.20}$$

Note that, in general, from (2.19) and (2.20), $k^* \neq \hat{k}$, i.e. the competitive equilibrium steady state is in general not socially optimal, so this economy suffers from a dynamic inefficiency. There may be too little or too much capital in the steady state, depending on parameter values. That is, suppose $\beta = 1$ and $n = .3$. Then, if $\alpha < .103$, $k^* > \hat{k}$, and if $\alpha > .103$, then $k^* < \hat{k}$.

2.5 Discussion

The above example illustrates the dynamic inefficiency that can result in this economy in a competitive equilibrium.. There are essentially two problems here. The first is that there is either too little or too much capital in the steady state, so that the quantity of resources available to allocate between the young and the old is not optimal. Second, the steady state interest rate is not equal to n, i.e. consumers face the "wrong" interest rate and therefore misallocate consumption goods over time; there is either too much or too little saving in a competitive equilibrium.

The root of the dynamic inefficiency is a form of market incompleteness, in that agents currently alive cannot trade with the unborn. To correct this inefficiency, it is necessary to have some mechanism which permits transfers between the old and the young.

2.6 Government Debt

We did not cover this point in classes

One means to introduce intergenerational transfers into this economy is through government debt. Here, the government acts as a kind of financial intermediary which issues debt to young agents, transfers the proceeds to young agents, and then taxes the young of the next generation in order to pay the interest and principal on the debt.

Let B_{t+1} denote the quantity of one-period bonds issued by the government in period t. Each of these bonds is a promise to pay $1+r_{t+1}$

units of consumption goods in period $t+1$. Note that the interest rate on government bonds is the same as the rental rate on capital, as must be the case in equilibrium for agents to be willing to hold both capital and government bonds. We will assume that

$$B_{t+1} = bL_t, \tag{2.21}$$

where b is a constant. That is, the quantity of government debt is fixed in per-capita terms. The government's budget constraint is

$$B_{t+1} + T_t = (1 + r_t)B_t, \tag{2.22}$$

i.e. the revenues from new bond issues and taxes in period t, T_t, equals the payments of interest and principal on government bonds issued in period $t-1$.

Taxes are levied lump-sum on young agents, and we will let τ_t denote the tax per young agent. We then have

$$T_t = \tau_t L_t. \tag{2.23}$$

A young agent solves

$$\max_{s_t} u(w_t - s_t - \tau_t, (1 + r_{t+1})s_t),$$

where s_t is savings, taking the form of acquisitions of capital and government bonds, which are perfect substitutes as assets. Optimal savings for a young agent is now given by

$$s_t = s(w_t - \tau_t, r_{t+1}). \tag{2.24}$$

As before, profit maximization by the firm implies (2.11) and (2.12).

A competitive equilibrium is defined as above, adding to the definition that there be a sequence of taxes $\{\tau_t\}_{t=0}^{\infty}$ satisfying the government budget constraint. From (2.21), (2.22), and (2.23), we get

$$\tau_t = \left(\frac{r_t - n}{1 + n}\right) b \tag{2.25}$$

The asset market equilibrium condition is now

$$k_{t+1}(1 + n) + b = s(w_t - \tau_t, r_{t+1}), \tag{2.26}$$

that is, per capita asset supplies equals savings per capita. Substituting in (2.26) for w_t, τ_t, and r_{t+1}, from (2.11), we get

$$k_{t+1}(1+n)+b = s\left(f(k_t) - k_t f'(k_t) - \left(\frac{f'(k_t)-n}{1+n}\right)b, f'(k_{t+1})\right) \quad (2.27)$$

We can then determine the steady state capital-labor ratio $k^*(b)$ by setting $k^*(b) = k_t = k_{t+1}$ in (2.27), to get

$$k^*(b)(1+n)+b = s\left(f(k^*(b)) - k^*(b)f'(k^*(b)) - \left(\frac{f'(k^*(b))-n}{1+n}\right)b, f'(k^*(b))\right) \quad (2.28)$$

Now, suppose that we wish to find the debt policy, determined by b, which yields a competitive equilibrium steady state which is socially optimal, i.e. we want to find $\hat{b}$ such that $k^*(\hat{b}) = \hat{k}$. Now, given that $f'(\hat{k}) = n$, from (2.28) we can solve for $\hat{b}$ as follows:

$$\hat{b} = -\hat{k}(1+n) + s\left(f(\hat{k}) - \hat{k}n, n\right) \quad (2.29)$$

In (2.29), note that $\hat{b}$ may be positive or negative. If $\hat{b} < 0$, then debt is negative, i.e. the government makes loans to young agents which are financed by taxation. Note that, from (2.25), $\tau_t = 0$ in the steady state with $b = \hat{b}$, so that the size of the government debt increases at a rate just sufficient to pay the interest and principal on previously-issued debt. That is, the debt increases at the rate n, which is equal to the interest rate. Here, at the optimum government debt policy simply transfers wealth from the young to the old (if the debt is positive), or from the old to the young (if the debt is negative).

2.6.1 Example

Consider the same example as above, but adding government debt. That is, $u(c^y, c^o) = \ln c^y + \beta \ln c^o$, and $F(K, L) = \gamma K^\alpha L^{1-\alpha}$, where $\beta > 0$, $\gamma > 0$, and $0 < \alpha < 1$. Optimal savings for a young agent is

$$s_t = \left(\frac{\beta}{1+\beta}\right)(w_t - \tau_t). \quad (2.30)$$

Then, from (2.16), (2.17), (2.27) and (2.30), the equilibrium sequence $\{k_t\}_{t=0}^{\infty}$ is determined by

$$k_{t+1}(1+n)+b=\left(\frac{\beta}{1+\beta}\right)\left[(1-\alpha)\gamma k_t^{\alpha}-\frac{(\alpha\gamma k_t^{\alpha-1}-n)b}{1+n}\right],$$

and the steady state capital-labor ratio, $k^*(b)$, is the solution to

$$k^*(b)(1+n)+b=\left(\frac{\beta}{1+\beta}\right)\left[(1-\alpha)\gamma\left(k^*(b)\right)^{\alpha}-\frac{(\alpha\gamma\left(k^*(b)\right)^{\alpha-1}-n)b}{1+n}\right]$$

Then, from (2.29), the optimal quantity of per-capita debt is

$$\begin{aligned}\hat{b} &= \left(\frac{\beta}{1+\beta}\right)(1-\alpha)\gamma\left(\frac{\alpha\gamma}{n}\right)^{\frac{\alpha}{1-\alpha}}-\left(\frac{\alpha\gamma}{n}\right)^{\frac{1}{1-\alpha}}(1+n)\\ &= \gamma\left(\frac{\alpha\gamma}{n}\right)^{\frac{\alpha}{1-\alpha}}\left[\frac{\beta(1-\alpha)}{1+\beta}-\frac{\alpha}{n}\right].\end{aligned}$$

Here note that, given γ, n, and β, $\hat{b}<0$ for α sufficiently large, and $\hat{b}>0$ for α sufficiently small.

2.6.2 Discussion

The competitive equilibrium here is in general suboptimal for reasons discussed above. But for those same reasons, government debt matters. That is, Ricardian equivalence does not hold here, in general, because the taxes required to pay off the currently-issued debt are not levied on the agents who receive the current tax benefits from a higher level of debt today. Government debt policy is a means for executing the intergenerational transfers that are required to achieve optimality. However, note that there are other intergenerational transfer mechanisms, like social security, which can accomplish the same thing in this model.

2.7 References

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